System Parameter and State Estimator over Unknown Linear Systems

Xuda Ding, Jianping He, Cailian Chen and Xinping Guan

***Abstract*—System model parameter and state estimations are classical problems for control theory. Most of the existing methods have promising results under the condition that the prior knowledge of the system model is known (system structure is given). However, in practical applications, the accuracy of prior knowledge degenerates over time, which causes identification failure. Besides, the estimation error bounds are needed for most of the low-cost model-based controller design. In this paper, we aim to design a stable parameter and state estimator over the unknown linear system (i.e., without prior knowledge), via a suitable feedback gain and iterative adjustment procedure and determine the error bounds. The insights of the proposed design lie in: i) the operation data can reflect the internal parameters of the model and provide a source for preliminary estimation; ii) the error bounds of estimated parameters and states can be determined based on mathematical analysis of the regression procedure. Specifically, the support vector regression (SVR) is used to estimate system parameters, and the feedback gain is designed to guarantee bounded-input, bounded-output (BIBO) stability. Then, the error bounds are given, and an adaptive adjustment procedure is adapted to increase the estimation accuracy. Finally, numerical simulations demonstrate the relative estimation error of the proposed estimator is less than 3.15%.**

I. INTRODUCTION

System dynamic parameter estimation has been studied for decades [1–11], which provides basic knowledge for the design of model-based controller and state estimator [12–14]. Conventional system identification methods are estimation techniques based on operation data, such as prediction error method [1, 2] and instrumental variable method [1, 3]. Later, subspace model identification (SMI) was proposed to improve the performance of the conventional identification methods and estimate the state-space model directly [4–8]. However, the accuracy of the identification methods depends on model type and system order selection. In practical applications, the model type and order are usually hard to determine, and the mismatch leads to the failure of identification. Besides, the aforementioned methods have few works on estimation error bound analysis, which causes difficulty for controller design.

Another technique to identify the system model is the adaptive observer design (a.k.a. data-driven observer design), which provides estimation error bound. Data-driven observer design has been adapted to different types of system, including single-input, single-output (SISO) linear time-invariant (LTI) system and multiple-inputs, multiple-outputs LTI system [9–11], and shown promising results under the assumption that system state matrix is an observer canonical form (or transformed one). When the system model structure is unknown, an adaptive observer cannot be used to identify the system.

In recent years, with the development of artificial intelligence and machine learning, lots of methods of data-driven modeling have been proposed, such as artificial neural networks (ANNs) [15] and support vector machines (SVMs) [16]. Further, characterized by strong abilities of self-learning and adaptivity, adaptive dynamic programming using reinforcement learning (RL) has demonstrated the capability to find the optimal control policy and solve the Bellman equation in a practical way [17–20]. However, the previous methods suffer from i) modeling is hard to regress an accurate model based on limited data, ii) the estimation bounds of the parameters and states are hard to analyze. In application, sensors degrade due to environmental effects such as vibrations and temperature fluctuations. Specifically, when the sensor fails, the valid data will be lost. Further, state-based or model-based controllers need bounds to be designed. Fortunately, nowadays, some studies focus on identifying system dynamics and topology based on limited observed data [21, 22]. They showed the potential for data to reflect the internal structure of the system, although the estimated parameters deviate from actual values.

Therefore, based on previous studies, we discover that: i) the preliminary system structure can be referenced by learning method based on a limited number of operation data; ii) the estimation error bound of parameter based on support vector regression (SVR) can be determined by matrix transforming and inequality scaling; iii) the design of state observer with a suitable observer (feedback) gain can guarantee the stability of the estimator and analyze the estimated states error bound. These discoveries provide theoretical feasibility to solve the drawbacks of the previous studies.

In this paper, we designed an adaptive parameter and state estimator with error bound over an unknown linear system. First, The SVR is used to form the system structure and estimate the parameters of the system. Then, the estimation error bounds of parameters are analyzed, and the observer gain is designed according to the bounds and guarantees bounded-input, bounded-output (BIBO) stability of the estimator. Further, the error bounds of the estimated system states are given. Finally, an adaptive adjustment procedure is proposed to adjust

[1]The authors are with Department of Automation, Shanghai Jiao Tong University, Key Laboratory of System Control and Information Processing, Ministry of Education of China and Shanghai Engineering Research Center of Intelligent Control and Management, Shanghai 200240, China. E-mails: {dingxuda, jphe, cailianchen, xpguan}@sjtu.edu.cn

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the parameters to increase the accuracy of the estimations at a low computational cost. The main challenges for this problem are i) the system is unknown (without prior knowledge); ii) the error bound of the estimation based on the data with noise need to be solved; iii) the choice of feedback gain and the design of the adjust procedure to make the observer stable and accurate. The main contributions are summarized as follows

- The combination of the learning method and feedback control provides preliminary structure and parameter estimation over an unknown system, which simplifies the assumption of needing prior system structure knowledge for conventional system identification.
- The estimation error bounds of parameters and system states are shown in parsed expressions, which can be further used for state-based and model-based controller design. Even a tighter bound is given under the condition of the state matrix being Jordan form.
- The design of the observer gain and the iterative adjustment method further improve the estimation accuracy while keeping the stability of the estimator. The following output of the system is used to adjust the estimator continuously, even if the state is no longer measured.

The rest of this paper is organized as follows. Section II gives the mathematical background and formulates the observer design problem over the unknown linear system. The SVR-based regression method, the error bound of the regressed state matrix, the design of observer gain, and the iterative adjustment procedure are presented in Sections III. Numerical simulation is conducted in Section IV. Conclusions and future works are drawn in Section V.

Notation: The scripts $\|\cdot\|$, $\|\cdot\|_2$ and $\|\cdot\|_{\mathrm{F}}$ represent norm, 2-norm and Frobenius norm function, respectively.

II. Preliminary and Problem Formulation

A. Preliminaries

First, we introduce preliminaries of SISO LTI system described in the state-space formula and classical Luenberger state observer. The SISO LTI system is described as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \omega, \\ y = \mathbf{C}\mathbf{x}, \end{cases} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the state matrix, $\mathbf{C} \in \mathbb{R}^{1\times n}$ is the output matrix, $\mathbf{B} \in \mathbb{R}^{n\times 1}$ is the input matrix. $\omega_i \sim N(0, \delta_i)$ is the noise, $\mathbf{x}$ is the state variable, y is the output variable, u is the bounded input where $\|u\| \le \beta_1 < \infty$. Classical Luenberger state observer of the LTI system is described by [23],

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{L}\left(y - \tilde{y}\right) + \mathbf{B}u, \\ \tilde{y} = \mathbf{C}\tilde{\mathbf{x}}, \end{cases} \tag{2}$$

where $\tilde{\mathbf{x}}$ and $\tilde{y}$ are the estimations of $\mathbf{x}$ and y, respectively. The system satisfies observability, which means the eigenvalues of the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ is designed arbitrarily by appropriate choice of the observer gain $\mathbf{L}$.

Then, we introduce the SVR. Let a data set be composed by N inputs $\mathbf{z}_i \in \mathbb{R}^{1\times d}, \forall i \in \{1, \cdots, N\}$, and their correspondent outputs $f(\mathbf{z}_i) \in \mathbb{R}$. f is a linear function as

$$f(\mathbf{z}_i) = \mathbf{w}\mathbf{z}_i + b, \tag{3}$$

where $\mathbf{w} \in \mathbb{R}^{d\times 1}$ is the weight vector, and b is a bias. The SVR finds optimal $\mathbf{w}$ and b for (3) such that

$$\min \sum_{i=1}^{N} |f(\mathbf{z}_i) - \mathbf{w}\mathbf{z}_i - b|, \tag{4}$$

and then reformulate the problem into convex optimization [16]. Note that the system is a linear model with noise, which is similar with (3). Thus, the SVR is a suitable tool for regressing state matrix in observer design.

B. Problem Description

In this paper, we mainly focus on the system dynamic with strong physical meaning, such as speed, voltage, temperature. When the measure points are set properly, the state variable $\mathbf{x}$ is measured by the equipment. Further, when sampling frequency is high enough, satisfies Shannon sampling condition, and $\dot{\mathbf{x}} \sim \Delta\mathbf{x}/\Delta t$, where Δt is a small sampling interval, thus, $\dot{\mathbf{x}}$ is obtained according to $\mathbf{x}$.

In many practical control systems, the input u can be controlled manually; the influence of input on the state variables $\mathbf{B}$ can be determined; the output transfer matrix $\mathbf{C}$ is designed by humans [24]. For example, the input force of the motion control system is manipulated manually, the impact of force on the state variables is measurable, and the output based on the state variables is designed manually. Similar systems broadly exist, such as electrical circuit, temperature control, liquid level control. These systems are easy for human intervention, which brings convenience for learning and studying. Therefore, based on the background, we introduce the following assumptions for the system (1) to formulate the problem.

Assumption 1. *$\mathbf{A}$ is stable, $\mathbf{B}$ and $\mathbf{C}$ are known, and u can be manipulated manually.*

Assumption 2. *$\mathbf{x}$ is measured during a period of time.*

Remark 1. *The proposed estimator is used for a stable system.*

Then, the goal of this paper is to design an effective parameter and state estimator based on limited observation of $\mathbf{x}$, the observation of y. To solve this problem, the following three issues need to be addressed.

i) Estimate parameters in $\mathbf{A}$ based on limited observation by SVR, and determine the regression error of $\hat{\mathbf{A}}$.
ii) Design observer gain $\hat{\mathbf{L}}$ to guarantee stability of the observer and estimate state variables.
iii) Design a procedure to adjust $\hat{\mathbf{A}}$ only based on y and improve the accuracy of the estimator.

Mathematically, $\hat{\mathbf{A}}$ is obtained from the regression of $\mathbf{A}$ based on the limited observation of $\mathbf{x}$. Then, observer gain $\hat{\mathbf{L}}$ is designed according to $\hat{\mathbf{A}}$. Hence, the parameter estimation and the proposed observer is

$$\min \left\|\hat{\mathbf{A}} - \mathbf{A}\right\|_{\mathrm{F}}, \tag{5a}$$

$$\text{s.t. } \dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\mathbf{x} + \hat{\mathbf{L}}\left(y - \hat{y}\right) + \mathbf{B}u, \tag{5b}$$

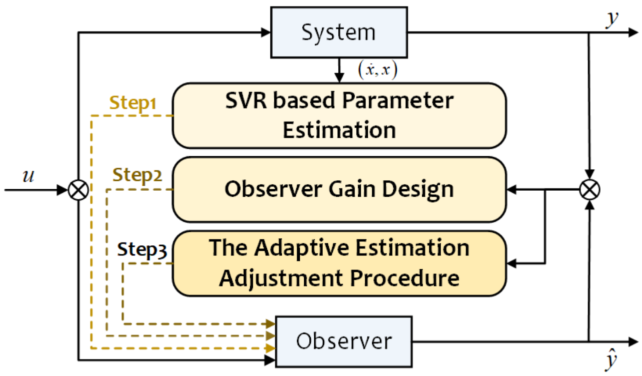


Fig. 1: The design of proposed estimator structure

where $\hat{\mathbf{x}}$ and $\hat{y}$ are the estimations of $\mathbf{x}$ and y, respectively, $\hat{\mathbf{L}}$ is the designed observer gain. Let $\hat{\mathbf{A}} = \mathbf{A} + \mathbf{\Delta A}$, and $\Delta\mathbf{A}$ is the regression error. Here due to $\hat{\mathbf{A}} \neq \mathbf{A}$, our goal is to design $\hat{\mathbf{L}}$ such that the eigenvalues of $\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}$ lie in left half plane, and we have state estimation as

$$\min \sup \lim_{t\to\infty} \|\mathbf{e}(t)\|_2 < \infty, \tag{6a}$$

$$\text{s.t. } \dot{\mathbf{e}} = (\mathbf{A} - \hat{\mathbf{L}}\mathbf{C})\mathbf{e} + \Delta\mathbf{A}\hat{\mathbf{x}} - \omega. \tag{6b}$$

where $k \in \mathbb{R}$, $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$. Then, it is needed to design iterative adjustment procedure for $\hat{\mathbf{x}}$, to increase the accuracy of estimations, as

$$\|\hat{\mathbf{A}} - \mathbf{A}\|_{\mathrm{F}} \xrightarrow[t\to\infty]{\text{as}} 0, \qquad \|\mathbf{e}(t)\|_2 \xrightarrow[t\to\infty]{\text{as}} 0. \tag{7}$$

III. Main Results

Before giving the main results and the proposed methods, we introduce the framework of this paper briefly (in Fig. 1). First, we use SVR as a regression procedure to regress $\mathbf{A}$ based on $\mathbf{x}$ and $\dot{\mathbf{x}}$ during $t_0 - t_f$, and get estimated $\hat{\mathbf{A}}$. Then, the observer gain $\hat{\mathbf{L}}$ is designed based on the estimated $\hat{\mathbf{A}}$ to keep the observer stable. After t_f, the iterative adjustment procedure is designed to adjust the parameters in $\hat{\mathbf{A}}$ based on the bias between actual y and predicted output of the observer, to improve the accuracy of the observer.

A. SVR-based Parameter Estimate

The SVR regression for state matrix is based on m times sampling of $\mathbf{x}$ in (1) from t_0 to t_f. The samplings compose the training data set $\{\dot{\mathbf{X}} \in \mathbb{R}^{n\times m}, \mathbf{X} \in \mathbb{R}^{n\times m}\}$, where $\dot{\mathbf{X}}$ is obtained according to $\mathbf{X}$. By comparing (1) and (3), $\mathbf{B}u$ is the only difference in the structure. The output of the SVR should be $\dot{\mathbf{x}} - \mathbf{B}u$. Since the $\mathbf{B}u$ is known, considering this term as a constant does not influence the analysis. Thus, it is ignored in the following of this subsection for convenience.

Based on $\{\dot{\mathbf{X}}, \mathbf{X}\}$ and using (3), it has

$$\dot{\mathbf{X}}_i = \hat{\mathbf{A}}_i\mathbf{X} + b, \tag{8}$$

To get a optimal $\hat{\mathbf{A}}$ and b, the following optimization problem is construct according to SVR method [16].

$$\min_{\hat{\mathbf{A}}_i, b, \xi^+, \xi^-} f(\hat{\mathbf{A}}_i, \xi^+, \xi^-) = \frac{1}{2}\left\|\hat{\mathbf{A}}_i\right\|^2 + c\sum_{j=1}^{m}(\xi_j^+ + \xi_j^-), \tag{9a}$$

$$\text{s.t.} \quad \dot{\mathbf{X}}_{i,j} - \mathbf{X}_j \le \varepsilon + \xi_i^+, \tag{9b}$$

$$\mathbf{X}_j - \dot{\mathbf{X}}_{i,j} \le \varepsilon + \xi_i^-, \tag{9c}$$

$$\xi_j^+ \ge 0, \xi_j^- \ge 0 \tag{9d}$$

where ξ^+, ξ^- are off-margin points and ϵ is the margin. Further, Lagrangian dual method and KKT condition are used to solve (9), and the dual function is constructed as

$$\max_{\alpha^+\ge 0, \alpha^-\ge 0} \sum_{j=1}^{m}\left[(\varepsilon - \dot{\mathbf{X}}_{i,j})\alpha_j^+ + (\varepsilon + \dot{\mathbf{X}}_{i,j})\alpha_j^-\right] - \frac{1}{2}\sum_{i=1,j=1}^{m}(\alpha_j^+ - \alpha_j^-)(\alpha_i^+ - \alpha_i^-)\mathbf{X}_j\mathbf{X}_i. \tag{10}$$

By solving the convex optimization problem, the Lagrangian parameters α^+, α^- and ϵ are obtained. Thus, $\hat{\mathbf{A}}_i$ is the regression result with error $\Delta\hat{\mathbf{A}}_i$ based on different training set, i.e., $\hat{\mathbf{A}}_i = \mathbf{A}_i + \Delta\mathbf{A}_i$.

Then, the error bound of $\hat{\mathbf{A}}$ is analyzed. Since SVR is a convex optimization, the optimal result can be obtained based on suitable data set. Thus, we have the reason to suppose $\{\mathbf{X}^* \in \mathbb{R}^{n\times l}, \dot{\mathbf{X}}_i^* \in \mathbb{R}^{1\times l}\}$ is the minimum sample set to obtain the actual value of $\mathbf{A}_i$. Then, Theorem 1 is given to determine the error of $\hat{\mathbf{A}}_i$.

Theorem 1. *Suppose there exists $\mathbf{M} \in \mathbb{R}^{l\times m}$, such that $\mathbf{X} = \mathbf{X}^*\mathbf{M}$. Then, we have*

$$\left(1 - 3\sqrt{l/m}\right)\left\|\hat{\mathbf{A}}_i\right\|_2^2 \le \|\mathbf{A}_i\|_2^2. \tag{11}$$

Note that the ratio of $\hat{\mathbf{A}}_i$ and $\mathbf{A}_i$ is given in the Theorem 1. Since $\hat{\mathbf{A}}_i$ is determined from the regression, the bound of $\Delta\mathbf{A}_i$ is not difficult to obtain from this ratio.

According to Theorem 1, the inequality solution for every elements in $\Delta\mathbf{A}$ and the range of $\mathbf{A}_{i,j}$ are

$$\frac{2\left(1 - 3\sqrt{l/m}\right) - 1}{1 - \left(1 - 3\sqrt{l/m}\right)}\Delta\mathbf{A}_{i,j}{}^2 + 2\Delta\mathbf{A}_{i,j}\hat{\mathbf{A}}_{i,j} - \hat{\mathbf{A}}_{i,j}^2 \le 0. \tag{12}$$

$$\mathbf{A}_{i,j} \in \left[\hat{\mathbf{A}}_{i,j} - h(\hat{\mathbf{A}}_{i,j}, l, m), \hat{\mathbf{A}}_{i,j} - g(\hat{\mathbf{A}}_{i,j}, l, m)\right]. \tag{13}$$

where the function $g(\hat{\mathbf{A}}_{i,j}, l, m)$ and $h(\hat{\mathbf{A}}_{i,j}, l, m)$ is used to describe the lower bound and the upper bound based on (12), respectively. And the the condition of the solution exists is that $3\sqrt{l/m} < 0.5$.

To simplify the mathematical expressions of the following context, we use $G_{i,j}$ and $H_{i,j}$ instead of $\hat{\mathbf{A}}_{i,j} - h(\hat{\mathbf{A}}_{i,j}, l, m)$ and $\hat{\mathbf{A}}_{i,j} - g(\hat{\mathbf{A}}_{i,j}, l, m)$, respectively.

Note that $\hat{\mathbf{A}}_i$ is one row of $\hat{\mathbf{A}}$. According to the condition of homogeneous equations having unique solution, $\mathbf{A}_i$ has a minimum optimal sample set with $l = n$, where $n = \mathrm{Rank}(\mathbf{A})$. However, it is not certain that $\{\mathbf{X}^*, \dot{\mathbf{X}}^*\}$ is the minimum for $\hat{\mathbf{A}}$ regression, exists during the observation of $\mathbf{x}$. Thus, the conservative estimation of $\{\mathbf{X}^*, \dot{\mathbf{X}}^*\}$ is composed by all minimum optimal row sets, and $l = n^2$. Fortunately, the following condition provides a minimum optimal sample set $\{\mathbf{X}^*, \dot{\mathbf{X}}^*\}$ for the regression of $\mathbf{A}$.

Corollary 1. *If $\mathbf{A}$ is Jordan matrix. Jordan block $J_i \neq J_j$, and $\mathbf{x} \neq 0$. Then, we have*

$$\left(1 - 3\sqrt{n/m}\right) \left\|\hat{\mathbf{A}}_i\right\|_2^2 \leq \|\mathbf{A}_i\|_2^2 . \tag{14}$$

Compares to Theorem 1, the ratio parameter is determined by $n = \mathrm{Rank}(\mathbf{A})$, where $n \leq l$. Thus, a tighter bound is get based on the same number of the samplings.

B. Observer Gain Design and State Estimate

After having $\hat{\mathbf{A}}$, observer gain $\hat{\mathbf{L}}$ is designed to make $\lim_{t\to\infty} \|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\|_2 \to k$. The main difficulty is how to use the bound given in Theorem 1, to design the observer gain and keep the system stable. Meanwhile, determining the bound of the $\mathbf{x}$ bias of the observer is also challenging. Theorem 2 is given for feedback gain design.

Theorem 2. *Suppose (13) holds true, the stable feedback gain is designed as*

$$\begin{cases} G_{i,i} - \hat{\mathbf{L}}_i\mathbf{C}_i < 0, \\ \sum\limits_{i\neq j, j=1}^{j=n} \max|\mathbf{U}_{ij}| \leq \left|G_{i,i} - \hat{\mathbf{L}}_i\mathbf{C}_i\right|. \end{cases} \tag{15}$$

where $\mathbf{U}_{ij} = \left[H_{i,j} - \hat{\mathbf{L}}_i\mathbf{C}_j, G_{i,j} - \hat{\mathbf{L}}_i\mathbf{C}_j\right]$. The observer with the designed gain guarantees the stable state estimation.

Proof. Consider (6b), due to the $\Delta\mathbf{A}\hat{\mathbf{x}}$ and the noise ω existing, BIBO stability stands if a) $\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}$ is designed to the left half plane and b) $\hat{\mathbf{x}}$ has bound (for the noise is zero-mean value, this term is ignored). The first step is designing a conservative observer gain to guarantee the stabilization of the observer.

It gets $\mathbf{A}_{i,j} - \hat{\mathbf{L}}_i\mathbf{C}_j \in \left[H_{i,j} - \hat{\mathbf{L}}_i\mathbf{C}_j, G_{i,j} - \hat{\mathbf{L}}_i\mathbf{C}_j\right]$ according to (13). Based on the Gershgorin circle theorem, the rightmost largest disc of ith eigenvalue of $\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}$ is

$$\left|z - (G_{i,i} - \hat{\mathbf{L}}_i\mathbf{C}_i)\right| \leq \sum_{i\neq j, j=1}^{j=n} \max|\mathbf{U}_{ij}|, \tag{16}$$

where $i \in \{1, 2, \cdots, n\}$. To guarantee the stability, every disc is conservatively set at the left half plane. Theorem 2 stands. □

With the design based on Theorem 2, $\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}$ is stable. Next, we analyze the bounds of $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}} - \mathbf{x}$. With $\|u\| \leq \beta_1 < \infty$, it has the bound of $\hat{\mathbf{x}}$ and the bound of $\hat{\mathbf{x}} - \mathbf{x}$

$$\|\hat{\mathbf{x}}\|_2 = \int_{t_0}^{t} \left|e^{\hat{\mathbf{A}}(t-\tau)}B\right| |u(\tau)|\, d\tau \leq \beta_x\beta_1, \tag{17a}$$

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_2 = \int_{t_0}^{t} \left|e^{(\mathbf{A}-\hat{\mathbf{L}}\mathbf{C})(t-\tau)}\Delta\mathbf{A}\right| \|\hat{\mathbf{x}}\|_2 d\tau \\ \leq \beta_e\beta_x\beta_1 < \infty. \tag{17b}$$

At this point, the BIBO stability stands with properly designed $\hat{\mathbf{L}}$.

Remark 2. *The design is conservative, when the Gershgorin circle of $\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}$ is too large to adjust, there is solution for (15). Nevertheless, the solution is sufficient for stability.*

Remark 3. *When the state transfer matrix satisfies Corollary 1, the gain is chosen precisely, for a) only diagonal terms of $\hat{\mathbf{A}}$ are with error; b) the error bound is tight with n. Further, when the matrix is diagonal, the radius of the Gershgorin disc is determined by $\hat{\mathbf{L}}$. Taking $\hat{\lambda}$ and $\Delta\lambda$ are the eigenvalues of $\hat{\mathbf{A}}$ and $\Delta\mathbf{A}$, respectively. From (14), it has*

$$\frac{2\left(1 - 3\sqrt{\frac{n}{m}}\right) - 1}{1 - \left(1 - 3\sqrt{\frac{n}{m}}\right)}\Delta\lambda_i^2 + 2\Delta\lambda_i\hat{\lambda}_i - \hat{\lambda}_i^2 \leq 0, \tag{18}$$

where $i \in \{1, 2, \cdots, n\}$. The observer gain $\hat{\mathbf{L}}$ is designed according to (15):

$$\begin{cases} \hat{\lambda}_i - g(\hat{\lambda}_i, l, m) - \hat{\mathbf{L}}_i\mathbf{C}_i < 0 \\ \sum\limits_{i\neq j, j=1}^{j=n} \left|\hat{\mathbf{L}}_i\mathbf{C}_j\right| \leq \left|\hat{\lambda}_i - g(\hat{\lambda}_i, l, m) - \hat{\mathbf{L}}_i\mathbf{C}_i\right| - \mathbf{A}_{i,i+1} \end{cases} \tag{19}$$

C. The Adaptive Estimation Adjustment Procedure

Since we have $\hat{\mathbf{A}}$ and $\hat{\mathbf{L}}$, the basic observer is operational. Next, we design an iterative adjustment procedure to adjust $\hat{\mathbf{A}}$ and track system output y. The predictive algorithm is adapted to predict the observer output without using the feedback of $y - \hat{y}$. The bias between the prediction and actual value reflects the error of $\hat{\mathbf{A}}$, which can be used to adjust $\hat{\mathbf{A}}$.

During the calculation process, the interval between sampling is Δt. According to (1) and (5), it has

$$y_{k+1} = \mathbf{C}\left(\mathbf{x}_k + \Delta t\left(\mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + \omega\right)\right), \tag{20}$$

$$\hat{y}_{k+1} = \mathbf{C}\left(\hat{\mathbf{x}}_k + \Delta t\left(\hat{\mathbf{A}}_k\hat{\mathbf{x}}_k + \mathbf{L}\left(y_k - \hat{y}_k\right) + \mathbf{B}u_k\right)\right). \tag{21}$$

The following procedures stand based on $\mathbf{x} \neq 0$ at t_f. $\hat{\mathbf{x}}_0$ is set to $\mathbf{x}$ at t_f as the initial value of the iterative procedure. The controllable input u guarantees $\mathbf{x} \neq 0$ as we mentioned above. $\hat{\mathbf{A}}_0$ is set to $\hat{\mathbf{A}}$.

Because output is the only variable obtained after t_f, so the goal is $\hat{y}_{k+1} = y_{k+1}$. According to (20) and (21), it gets

$$\begin{aligned} &\hat{y}_{k+1} - \Delta t\mathbf{C}\mathbf{L}\left(y_k - \hat{y}_k\right) - \Delta t\mathbf{C}\mathbf{B}u_k \\ &= (1 + \Delta t)\mathbf{C}\hat{\mathbf{A}}_k\hat{\mathbf{x}}_k = y_{k+1} - \Delta t\mathbf{C}\mathbf{B}u_k + \Delta t\mathbf{C}\omega \end{aligned} . \tag{22}$$

Let $\mathbf{C}(1 + \Delta t)\hat{\mathbf{A}}_k = \varpi_k$ and $y_{k+1} - \Delta t\mathbf{C}\mathbf{B}u_k = \theta_k$, thus, $\varpi_k\hat{\mathbf{x}}_k = \theta_k + \Delta t\mathbf{C}\omega$. To simplify the analysis, according to [25], the noise effect of the output is ignored during the iteration procedure. $\varpi_k\hat{\mathbf{x}}_k = \theta_k + \Delta t\mathbf{C}\omega$ is reformulated into an optimization problem [26], as

$$\min_{\varpi^{\mathrm{T}}} \left\{ \left\|\hat{\mathbf{x}}^{\mathrm{T}}\varpi^{\mathrm{T}} - \theta^{\mathrm{T}}\right\|^2 + \frac{\left\|\varpi^{\mathrm{T}}\right\|^2}{\sigma} \right\}. \tag{23}$$

where σ is the regularization parameter which keeps $\hat{\mathbf{x}}_k\hat{\mathbf{x}}_k^{\mathrm{T}}$ being non-singular.

Algorithm 1: Iterative Adjustment Algorithm

Input: $\{\mathbf{X}, \dot{\mathbf{X}}\}$, $\mathbf{x}_{t_f}$, y, u, $\mathbf{B}$, $\mathbf{C}$, Δt, $\hat{\mathbf{L}}$,$\mathbf{\Gamma}$.
Output: $\hat{\mathbf{x}}$, $\hat{y}$, $\hat{\mathbf{A}}$.

if *At the beginning of the second stage* **then**
  $\hat{\mathbf{A}}_0 \leftarrow$SVR based on $\{\mathbf{X}, \dot{\mathbf{X}}\}$.
end
if y_1 *exists* **then**
  $\theta_0 \leftarrow y_1 - \Delta t\mathbf{CB}u_0$;
  $\varpi_0^{\mathrm{T}} \leftarrow (\hat{\mathbf{x}}_0\hat{\mathbf{x}}_0^{\mathrm{T}} + \frac{\mathbb{I}}{\sigma})^{-1}\hat{\mathbf{x}}_0\theta_0^{\mathrm{T}}$;
  $\mathbf{P}_0 \leftarrow \left(\hat{\mathbf{x}}_0\hat{\mathbf{x}}_0^{\mathrm{T}} + \frac{\mathbb{I}}{\sigma}\right)^{-1}$.
end
while *During the second stage* **do**
  calculate (25a);
  calculate (25b);
  $\hat{\mathbf{A}}_{k+1} = \frac{\mathbf{C}^{-1}\varpi_{k+1}}{(1+\Delta t)}$, and satisfies the restraint condition;
  $\hat{\mathbf{x}}_{k+2} \leftarrow \hat{\mathbf{x}}_{k+1} + \Delta t\left(\hat{\mathbf{A}}_{k+1}\hat{\mathbf{x}}_{k+1} + \hat{\mathbf{L}}\left(y_{k+1} - \hat{y}_{k+1}\right) + \mathbf{B}u_{k+1}\right)$;
  $\hat{y}_{k+2} \leftarrow \mathbf{C}\left(\hat{\mathbf{x}}_{k+2}\right)$.
  Circular assignment.
  return $\hat{\mathbf{x}}$, $\hat{y}$, $\hat{\mathbf{A}}$.
end

At the initial stage of the iteration, the optimal solution is calculated by the following equation [27], as

$$\varpi_0^{\mathrm{T}} = (\hat{\mathbf{x}}_0\hat{\mathbf{x}}_0^{\mathrm{T}} + \frac{\mathbb{I}}{\sigma})^{-1}\hat{\mathbf{x}}_0\theta_0^{\mathrm{T}}. \tag{24}$$

Taking $\mathbf{P}_0 = \left(\hat{\mathbf{x}}_0\hat{\mathbf{x}}_0^{\mathrm{T}} + \frac{\mathbb{I}}{\sigma}\right)^{-1}$, the iteration procedure is described as follows [25].

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \frac{\mathbf{P}_k\hat{\mathbf{x}}_{k+1}\hat{\mathbf{x}}_{k+1}^T\mathbf{P}_k}{1 + \hat{\mathbf{x}}_{k+1}^T\mathbf{P}_k\hat{\mathbf{x}}_{k+1}}, \tag{25a}$$

$$\varpi_{k+1}^{\mathrm{T}} = \varpi_k^{\mathrm{T}} + \mathbf{P}_{k+1}\hat{\mathbf{x}}_{k+1}(\theta_{k+1} - \hat{\mathbf{x}}_{k+1}^{\mathrm{T}}\varpi_k^{\mathrm{T}}). \tag{25b}$$

Then, new observer state matrix is obtained as

$$\min_{\hat{\mathbf{A}}_{k+1}} \left\|\mathbf{\Gamma}(\hat{\mathbf{A}}_{k+1} - \hat{\mathbf{A}}_k)\right\|_{\mathrm{F}}, \tag{26a}$$

$$\text{s.t.} \quad \hat{\mathbf{A}}_{k+1} - \hat{\mathbf{A}}_0 \in \left[g(\hat{\mathbf{A}}_{i,j}, l, m), h(\hat{\mathbf{A}}_{i,j}, l, m)\right], \tag{26b}$$

where $\hat{\mathbf{A}}_{k+1} = \frac{\mathbf{C}^{-1}\varpi_{k+1}}{(1+\Delta t)}$ and $\mathbf{\Gamma}$ is the parameter matrix.

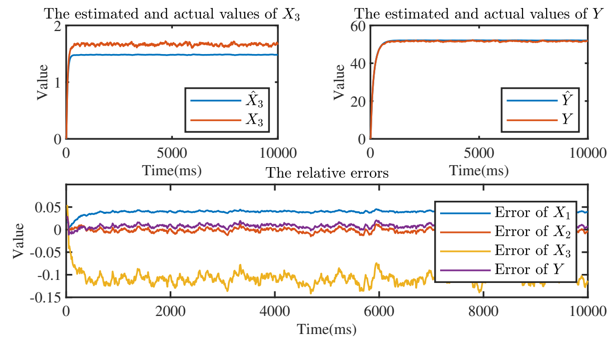
Note that $\mathbf{C}^{-1}\varpi_{k+1}$ has multiple solutions, $\mathbf{\Gamma}$ is designed to improve the convergence speed, and the restraint condition is designed according to (12) under Theorem 1 or (18) under Corollary 1. Algorithm 1 for iterative adjustment is presented.

IV. Numerical Simulation

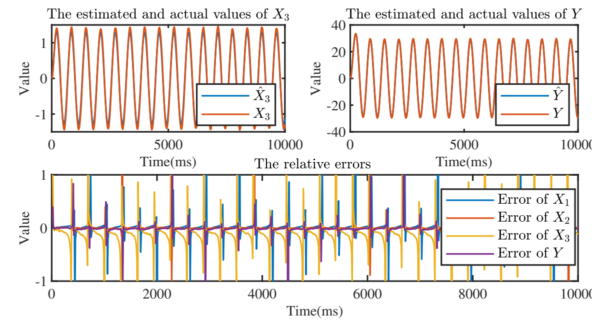
An observable LTI SISO system is with $\mathbf{A} = \mathrm{diag}\{-5, -10, -15\}$, $\mathbf{B} = \left[\ 15;10;5\ \right]$, $\mathbf{C} = \left[\ 234\ \right]$, $\omega \sim \mathbf{N}(\mathbf{0}, \mathbf{1})$.

A. Parameter estimate

To regress $\mathbf{A}$, we generate 150 non-zero vector samples of $\mathbf{x}$ and $\dot{\mathbf{x}}$. During the regression process, "simplex" algorithm is used for searching the regularization parameter in (9), "leave-one-out" method is used as an estimate



(a) The state estimation of step response



(b) The state estimation of sin response

Fig. 2: The state estimation without adjustment procedure

of the performance of the model. We get $\hat{\mathbf{A}}_0$ as $\hat{\mathbf{A}}_0 = \mathrm{diag}\{-4.78, -10.03, -16.95\}$.

B. State estimate

According to Corollary 1, $n = 3, m = 150$. The bound of $\Delta\lambda$ is determined by (18). We have $\Delta\lambda_1 = 0.22 \in [-2.21, 29.01], \Delta\lambda_2 = -0.03 \in [-4.63, 60.80]$, $\Delta\lambda_3 = -1.95 \in [-7.83, 102.78]$. It is obvious that the results of bound are correct.

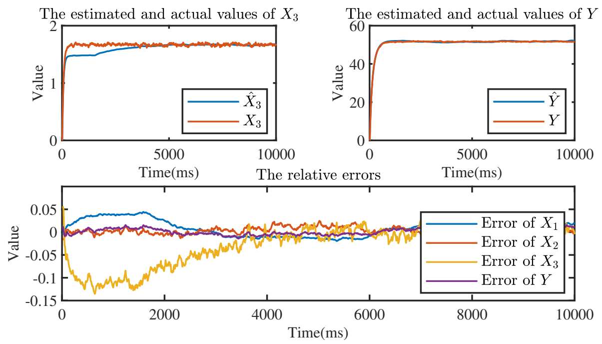
Note that as it is shown in Fig. 4, the range between upper and lower bound decreases as the number of the training sample increases. Thus, the performance of regression is better when more samples are used. The result is consistent with common knowledge. Then, we design the observer gain $\hat{\mathbf{L}}$ based on Algorithm 1 and have $\hat{\mathbf{L}} = [0.20; 0.60; 0.20]$ and $\mathrm{eig}(\mathbf{A} - \hat{\mathbf{L}}\mathbf{C}) = [-5.27; -11.53; -16.18]$.

C. Estimation without Adjustment Procedure

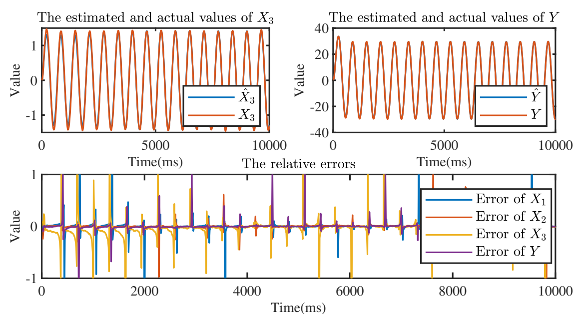
The simulation in this subsection is designed to show the influence caused by inaccurate $\hat{\mathbf{A}}$. The observer with $\hat{\mathbf{L}}$ is tested twice with the input of step and sine wave signals. The results are shown in Fig. 2(a) and Fig. 2(b).

In Fig. 2(a), X_3 has the largest error among the state variables. The reason for this is that a larger error of eigenvalue causes a larger error of the state variable. In the stable stage (after 4000ms), the maximum relative error of X_1, X_2 and X_3 are 4.5%, 1.5% and 14.2%, respectively. Fortunately, the error is bounded with a well-designed observer gain $\hat{\mathbf{L}}$, which makes the observer stable and $\hat{y}$ track y well.

Fig. 2(b) shows that the errors are not so obviously shown but still exist when the system is driven by sin input. The



(a) The state estimation of step response



(b) The state estimation of sin response

Fig. 3: The state estimation with adjustment procedure

reason is that the system never reaches a stable stage, and the observer gain drags the estimated state variables to actual value by the bias of $\hat{y} - y$.

D. *Estimation with Adjustment Procedure*

The following simulation is designed to show that the adjustment procedure ensures the estimated variables to track better and make the estimated matrix to be more accurate. We choose 1500 ms of the simulation in subsection C as the initial stage, to verify the performance improved by adjustment procedure. The results are shown in Fig. 3(a) and Fig. 3(b).

In Fig. 3(a), the estimated state variables begin to track the actual value after 1500 ms. Although there are fluctuations caused by noises in the system, the adjustment procedure guarantees the observer has a good performance. At the stable stage (after 4000ms), the maximum relative error of X_1, X_2 and X_3 are 2.0%, 2.5% and 3.2%, respectively. The final $\hat{\mathbf{A}}$ at 10000 ms in Fig. 2 is $\mathrm{diag}\{-4.92, -9.99, -15.26\}$.

In conclusion, the numerical simulations illustrate the effectiveness of the proposed methods.

V. CONCLUSIONS

In this paper, a system model parameters and state estimator is designed for the unknown linear system. The state matrix is regressed by SVR, and the regression error is determined by mathematical analysis. The observer gain is assigned based on the regression error and guarantees the BIBO stability of the observer. Finally, an adaptive estimation adjustment procedure is proposed to adjust the state matrix and reduce the errors of the estimated variables. The numerical simulations further verify the effectiveness of the proposed estimator.

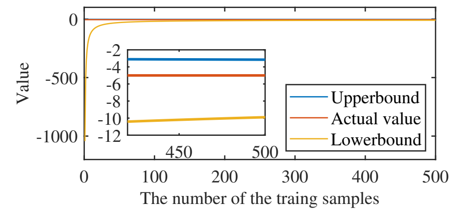


Fig. 4: The bound of estimate eigenvalue

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