Consensus Under Bounded Noise in Discrete Network Systems: An Algorithm With Fast Convergence and High Accuracy

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***Abstract*—Most existing works investigate consensus with noise following a certain distribution, e.g., Gaussian distribution, with fixed expectation and variance, which may not be satisfied in practical applications. This paper investigates the discrete system consensus under bounded noise, which is important and practical problem. We first provide necessary and sufficient conditions for the convergence of consensus under bounded noise. To be more general, we derive an analytical bound to show the max–min difference between the nodes' states when the general consensus algorithm converges to a stable state. Then, a novel consensus algorithm, fast consensus under bounded noise (FCBN), is proposed to eliminate the accumulative error caused by the bounded noise. It is proved that FCBN has a faster convergence speed and a higher consensus accuracy than general consensus algorithms. Extensive simulations demonstrate the effectiveness of the proposed algorithm.**

***Index Terms*—Bounded noise, consensus, fast convergence.**

I. Introduction

CONSENSUS refers to the action that autonomous agents/nodes reach an agreement regarding a certain opinion. Applications of consensus can be found in a variety of domains, e.g., time synchronization [2]–[5], coordination and cooperation [6]–[11], distributed estimation and optimization [12]–[14], sensor fusion [15], [16], distributed scheduling [17], and decision making [18]. Different consensus problems have been studied by researchers, e.g., average consensus [19]–[21], dynamic consensus [22], security of consensus [23]–[26], etc.

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Different types of noises, e.g., external disturbance, measurement error, communication delay, etc., are inevitable in a network [27]–[30]. The study of consensus with noise has attracted increasing attention, especially regarding the reliability of consensus [31]–[34] and the resilient control with consensus [35], [36]. For example, Hatano *et al.* [28] studied consensus with noisy interconnections, which is under the assumption that the noise follows a Gaussian distribution and the nodes are uniformly distributed. The result reveals that higher noise variance can lead to lower convergence rate. Recently, Rajagopal and Wainwright [33] considered consensus over noisy channels with zero mean random noise, and obtained almost sure convergence and predicted the error under noise when the topology of the network is known. The noisy environment may cause the consensus not be achieved completely [32]. The authors designed a controller to reduce the bound, and achieved a more accurate consensus. Hu *et al.* [21] investigated consensus of the leader–follower continuous-time system with noises and gave a sufficient condition to guarantee the consensus. Carli *et al.* [37] discussed the influence of noise on discrete average consensus, where the noise is assumed to be white with zero mean and bounded covariance. The authors concluded that consensus under such noise deviates from the average consensus by a value that increases linearly with the iteration time. Huang and Manton [31] proposed a stochastic approximation-type algorithm with a decreasing step size to establish mean square consensus and strong consensus. Liu *et al.* [34] proposed a protocol to achieve mean square consensus and strong consensus, where the noise has zero mean and uniformly bounded covariance. Noise has also been considered in quantized data and link failures [30] and estimation of parameters or signals [38], both of which assumed that the distribution of the noise is known.

Most existing works mainly focus on noise with a known distribution or with a known mean and covariance, and Gaussian distribution is widely used. However, some noises may not be random variables with fixed means and variances, e.g., the negative noise caused by clock skew degradation [3], measurement errors [39], and external disturbances [42]. To overcome this, a realistic noise model, bounded noise, i.e., the noise changes randomly between an upper and lower bound, is considered for consensus recently [39], [42], where some algorithms are designed to reduce the effect of bounded noise. Specifically, Garulli and Giannitrapani [39] to dynamically adjust the weight of the neighboring agents for the discrete average consensus. Wang and Liu [42] investigated the robust consensus, where the external disturbance is also viewed as bounded noise. They showed that robust consensus can be reached when the union of the collection of the interaction graphs across some times has a spanning tree. Although the proposed algorithm achieves general convergence under bounded noise, the convergence speed is slow. The conditions to guarantee convergence also rely on the network topology, such as whether the graph has a spanning tree. In this paper, we investigate how the bounded noise influences the convergence of discrete average consensus, and establish the relationship between the consensus accuracy and the bounds of the noise. By exploiting the principle that a bounded monotonic sequence must possess a limit and the idea of maximum consensus, we then design a novel algorithm to achieve fast and accurate consensus under bounded noise. Under the proposed algorithm, the states of nodes can increase monotonically due to the noise, but be bounded by a value, and thus can guarantee the realization of consensus. The main contributions of this paper are summarized as follows.

1) We study discrete average consensus under a practical noise model, bounded noise, where the noise is only bounded by a constant without any assumption of its distribution, which is general and challenging.
2) Necessary and sufficient conditions are provided for the system to achieve consensus under bounded noise. We further derive a theoretical gap, showing the deviation of the stable state under bounded noise from its true average value. We also analyze the relationship between the consensus accuracy and the noise bound.
3) A novel algorithm, named fast consensus under bounded noise (FCBN), is designed to protect consensus from the noise, according to which each node updates its own value based on the principle of maximum-value-based consensus at each iteration. It is proved that the algorithm has fast convergence speed as well as high consensus accuracy.

The remainder of this paper is organized as follows. Section II provides the system models and formulates the problem. Main results are presented in Section III and Section IV, among which the FCBN algorithm is proposed in Section IV. Section V verifies the main results and the performance of the proposed algorithm through extensive simulations. The conclusions are given in Section VI.

II. Problem Formulation

A. Network Model

Consider a network with N nodes denoted by $1, 2, \ldots, N$. An undirected graph $\mathcal{G}(t) = \{V, E(t)\}$ is used to represent the communication topology of the network, where V is the set of nodes and $E(t) \subset V \times V$ is the set of edges. The edge $(i, j) \in E(t)$ if and only if (iff) node i communicates with j at time t, and we say that node j is the neighbor of node i at time t. At each time t, the neighbor set of node i is defined as $\mathcal{N}_i(t) = \{j|(j, i) \in E(t), j \neq i\}$. The communication topology of a fixed network is denoted as $\mathcal{G} = \{V, E\}$. We assume that $\mathcal{G}(t) = \{V, E(t)\}$ is a connected graph, which is a common assumption [10]. Let d be the diameter of the graph $\mathcal{G}$. Assume that each node i starts with a scalar value $x_i(0)$ and let $\mathbf{X}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$ be the state vector of all nodes at time t.

Let each node i's 0-hop neighbor be node i itself. Define $\mathcal{D}_0^i$ as the 0-hop neighbor set of node i, and define $\mathcal{D}_K^i$ as the set of the K-hop neighbors of node i for $K = 1, 2, \ldots, N-1$. The graph $\mathcal{G}(t) = \{V, E(t)\}$ is assumed to be connected, thus we have $\mathcal{D}_K^i \subset V \backslash \{\mathcal{D}_0^i \cup \cdots \cup \mathcal{D}_{K-1}{}^i\}$ and $V = \mathcal{D}_0^i \cup \cdots \cup \mathcal{D}_{N-1}^i$.

B. Bounded Noise Model

In this paper, similar to the models proposed in [22], an important and realistic model, bounded noise, is used which is given by

$$x_i^+(t) = x_i(t) + m_i(t), i \in V \tag{1}$$

where $x_i(t)$ is the real state and $m_i(t)$ is a noise, satisfying

$$|m_i(t)| \leq \frac{1}{2}\delta \tag{2}$$

where δ is a positive bound. In (1), $x_i^+(t)$ is the broadcast state value of node i at time t. It should be noted that the disturbances, e.g., a bounded false data being injected by a misbehaving node in an unreliable network [43], can also be modeled by (1). For the ideal case or where the noise can be negligible, one has $x_k^+(t) = x_k(t)$, i.e., $m_k(t) \equiv 0$.

C. Discrete Consensus With Bounded Noise

We consider the discrete average consensus algorithm (DACA) described in [8] using model (1)

$$\begin{aligned} x_i(t+1) &= x_i^+(t) + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j^+(t) - x_i^+(t)) \\ &= \sum_{j=1}^{N} a_{ij}(t)x_j(t) + \sum_{j=1}^{N} a_{ij}(t)m_j(t) \end{aligned}$$

where $m_i(t)$ satisfies (2), $a_{ij}(t)$ is a positive weight for $j \in \mathcal{N}_i(t)$, and set $a_{ii}(t) = 1 - \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)$. Its matrix form is given by

$$\mathbf{X}(t+1) = \mathbf{A}(t)(\mathbf{X}(t) + \mathbf{M}(t)) \tag{3}$$

where $\mathbf{M}(t) = \{m_i\}_{N\times 1}$ and $A(t) = \{a_{ij}\}_{N\times N}$. If the noise $\mathbf{M}(t)$ is a random variable with $\mathbf{E}\{M(t)\} = 0$, Aysal and Barner [40] have proved that the consensus can be achieved with probability 1. If the noise $\mathbf{M}(t)$ is ignored, the following two assumptions guarantee the convergence of (3), which is an average consensus process [10]. For the simplicity, when noise cannot be ignored, we denote the updating process (3) as DACA under noise.

Assumption 1: There exists a positive constant α such that $a_{ii}(t) \geq \alpha$ and $a_{ij}(t) \in \{0\} \cup [\alpha, 1]$ for all i, j, t.

Assumption 2 (Balanced Communication): For any $t \geq 0$, $\mathbf{1}^T\mathbf{A}(t) = \mathbf{1}^T$ and $\mathbf{A}(t)\mathbf{1} = \mathbf{1}$, i.e., $\mathbf{A}(t)$ is doubly stochastic.[1]

Remark 1: Assumption 1 means that each node will use its neighbors' information to update its own state, and Assumption 2 guarantees that all nodes converge to the average value of their initial values, i.e., $\lim_{t\to\infty} x_i(t) = (1/N)\sum_{i=1}^N x_i(0) = \bar{x}$ for any i [8], [20]. When taking noise into consideration, however, the convergence may not be guaranteed as illustrated in the remainder of the paper.

If $\mathbf{M}(t) \equiv 0$, i.e., the noise is ignored, some promising works have been provided to guarantee the fast consensus under (3). For example, Xiao and Boyd [44] proposed a fast average consensus algorithm with linear optimization approach, which was further improved by Zhang and Chen [45] using predictive mechanism. If $\mathbf{M}(t)$ is assumed to be random white noise with zero mean and bounded variance, it is equivalent to the problem that was considered in [28], [37], and [38], in which the stability conditions and the influence of noise on the convergence have been obtained. The $\mathbf{M}(t)$ considered in our paper, however, may not be a white noise or even without fixed mean and variance, and the results in the existing literatures cannot be applied.

For bounded noises, there are at least three interesting problems. The first is under what conditions the network is still able to achieve consensus. The second is how the noise bound δ affects the consensus accuracy (i.e., the fluctuation bound of the stable state) of the consensus algorithm (3). And the third is how to mitigate the accumulation error for the stable state caused by bounded noise to achieve more accurate consensus. In this paper, we will tackle these three problems.

III. Average Consensus With Bounded Noise

In order to facilitate the presentation, define

$$H_t = \max_{i\in V} x_i(t), \quad h_t = \min_{i\in V} x_i(t), \quad M_t = \max_{i\in V} m_i(t)$$
$$m_t = \min_{i\in V} m_i(t), \quad D_t = H_t - h_t, \quad \Delta m_t = M_t - m_t.$$

Definition 1: Define consensus accuracy, η, satisfying

$$\lim_{t\to\infty} D_t \leq \eta, \eta \in [0, \infty]. \tag{4}$$

Consensus accuracy characterizes the upper bound of the fluctuation of maximum difference between nodes' states when a consensus algorithm converges to a stable state. We say the consensus is achieved completely, i.e., complete convergence, iff $\eta = 0$ in (4).

A. Necessary and Sufficient Conditions

By referring to [41, Th. 1], we give the following necessary condition for DACA to reach convergence in the presence of bounded noise.

Theorem 1 (Necessary Condition): Under Assumptions 1 and 2, if (3) achieves complete convergence, i.e., $\lim_{t\to\infty} D_t = 0$, then $\lim_{t\to\infty} \mathbf{M}(\mathbf{t}) = \mathbf{0}$, where $\mathbf{0}$ is a vector with all its entries being zero.

[1] A doubly stochastic matrix is a square matrix of non-negative real entries, with all the elements in each row (column) summed up to be 1.

Theorem 2 (Sufficient Condition): Under Assumptions 1 and 2, if

$$\sum_{t=0}^{\infty} |m_i(t)| \leq B, i \in V \tag{5}$$

where B is a constant bound, then using the algorithm (3), we have $\lim_{t\to\infty} x_i(t) = c, i \in V$, where c is a constant.

Proof: First, as the maximum eigenvalue of each $\mathbf{A}(t)$ equals to 1, we have

$$\left\| \left[\prod_{h=p}^{t} \mathbf{A}(h-1)\right] \mathbf{M}(p-1) \right\|_\infty \leq \left\| \prod_{h=p}^{t} \mathbf{A}(h-1) \right\|_\infty \|\mathbf{M}(p-1)\|_\infty \leq \|\mathbf{M}(p-1)\|_\infty. \tag{6}$$

For $\forall i \in V$, since $\sum_{t=0}^{\infty} |m_i(t)| \leq B$, we have the series $m_i(t)$ is absolute convergent, which implies that

$$\lim_{t\to\infty} \sum_{k=t}^{\infty} |m_i(k)| = 0.$$

Note that for $\forall i \in V$

$$-\sum_{k=t}^{\infty} |m_i(k)| \leq \sum_{k=t}^{\infty} m_i(k) \leq \sum_{k=t}^{\infty} |m_i(k)|.$$

Taking limitation on the above equation over t yields

$$\lim_{t\to\infty} \sum_{k=t}^{\infty} m_i(k) = 0, \forall i \in V. \tag{7}$$

Subsequently, it follows from (7) that $\lim_{t\to\infty} \sum_{k=t}^{\infty} \|\mathbf{M}(k)\|_\infty = 0$. Hence, from (6), we have

$$\lim_{k\to\infty} \sum_{p=k}^{\infty} \left\| \left[\prod_{h=p}^{\infty} \mathbf{A}(h-1)\right] \mathbf{M}(p-1) \right\|_\infty \leq \lim_{t\to\infty} \sum_{k=t}^{\infty} \|\mathbf{M}(k)\|_\infty = 0. \tag{8}$$

Meanwhile, it is clear that

$$\lim_{k\to\infty} \sum_{p=k}^{\infty} \left\| \left[\prod_{h=p}^{\infty} \mathbf{A}(h-1)\right] \mathbf{M}(p-1) \right\|_\infty \geq 0. \tag{9}$$

By combining (8) with (9), we have

$$\lim_{k\to\infty} \sum_{p=k}^{\infty} \left\| \left[\prod_{h=p}^{\infty} \mathbf{A}(h-1)\right] \mathbf{M}(p-1) \right\|_\infty = 0.$$

Hence, $\sum_{p=1}^{t}(\prod_{h=p}^{t} \mathbf{A}(h-1))\mathbf{M}(p-1)$ converges when $t \to \infty$. Note that each $\mathbf{A}(t)$ is a doubly stochastic matrix, thus

$$\lim_{t\to\infty} \sum_{p=1}^{k} \left[\prod_{h=p}^{t} \mathbf{A}(h-1)\right] \mathbf{M}(p-1) = c_1\mathbf{1}$$

for a given k, where c_1 is a constant, and $\lim_{t\to\infty} \prod_{p=1}^{t} \mathbf{A}(p-1) \mathbf{X}(0) = \bar{x}\mathbf{1}$.

From discrete consensus system (3), we obtain

$$\mathbf{X}(t) = \prod_{p=1}^{t} \mathbf{A}(p-1)\mathbf{X}(0) + \sum_{p=1}^{t} \left(\prod_{h=p}^{t} \mathbf{A}(h-1) \right) \mathbf{M}(p-1). \tag{10}$$

Therefore, it follows from (10) that $\lim_{t\to\infty} \mathbf{X}(t) = c\mathbf{1}$. ∎

Theorem 2 provides a sufficient condition for system (3) with bounded noise to achieve convergence. For example, when the noise caused by a damped external noise source, which satisfies (5), then (3) converges. As another example, for some cyber attacks, where the attackers are constrained by a power budget or are power-limited [23], the number of attacks is finite, which also satisfies the sufficient condition.

Although under the sufficient condition (5), consensus can be achieved completely for the discrete average consensus (3), there exists a gap between its stable state and its true average value, which is presented in the following theorem.

Theorem 3: Suppose (5) holds. Under Assumptions 1 and 2, using the algorithm (3), we have

$$\lim_{t\to\infty} \left| \frac{1}{N} \sum_{i=1}^{N} x_i(t) - \bar{x} \right| \le \lim_{t\to\infty} \sum_{p=0}^{t} \max_{i\in V} |m_i(p)| \le \mathrm{BN}. \tag{11}$$

Proof: It follows from Theorem 2 that system (3) converges under condition (5). From (10), we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t) &= \lim_{t\to\infty} \frac{1}{N} \mathbf{1}^T \prod_{p=1}^{t} \mathbf{A}(p-1)\mathbf{X}(0) \\
&\quad + \lim_{t\to\infty} \frac{1}{N} \mathbf{1}^T \sum_{p=1}^{t} \left[\prod_{h=p}^{t} \mathbf{A}(h-1) \right] \mathbf{M}(p-1) \\
&= \frac{1}{N} \sum_{i=1}^{N} x_i(0) \\
&\quad + \lim_{t\to\infty} \frac{1}{N} \mathbf{1}^T \sum_{p=1}^{t} \left[\prod_{h=p}^{t} \mathbf{A}(h-1) \right] \mathbf{M}(p-1) \\
&= \bar{x} + \lim_{t\to\infty} \frac{1}{N} \mathbf{1}^T \sum_{p=1}^{t} \left[\prod_{h=p}^{t} \mathbf{A}(h-1) \right] \mathbf{M}(p-1).
\end{aligned}$$

Then it follows from (5) and (6) that:

$$\begin{aligned}
&\lim_{t\to\infty} \left| \frac{1}{N} \sum_{i=1}^{N} x_i(t) - \bar{x} \right| \\
&\le \left\| \lim_{t\to\infty} \sum_{p=1}^{t} \left[\prod_{h=p}^{t} \mathbf{A}(h-1) \right] \mathbf{M}(p-1) \right\|_\infty \\
&\le \lim_{t\to\infty} \sum_{p=1}^{t} \left\| \left[\prod_{h=p}^{t} \mathbf{A}(h-1) \right] \mathbf{M}(p-1) \right\|_\infty \\
&\le \lim_{t\to\infty} \sum_{p=1}^{t} \|\mathbf{M}(p-1)\|_\infty \le \lim_{t\to\infty} \sum_{p=0}^{t} \max_{i\in V} |m_i(p)|.
\end{aligned}$$

∎

Since $|m_i(t)| \le (1/2)\delta$, $\sum_{p=0}^{t} \max_{i\in V} |m_i(p)|$ is monotonically nondecreasing and has an upper bound BN, $\sum_{p=0}^{t} \max_{i\in V} m_i(p)$ converges when $t \to \infty$. The upper bound BN is a conservative bound, as it is reached under the worst connectivity of the network and accompanied by the maximum noise satisfying (5). More details are presented in the simulation section.

B. Relationship Between Noise Bound and Accuracy

To reveal the influence of bounded noise on the stable state of consensus, we study the relationship between the noise bound and the consensus accuracy of (3) in this section, and get the following results.

Theorem 4: Under Assumptions 1 and 2, using the algorithm (3), we have

$$\lim_{t\to\infty} D_t \le \frac{\left(1+\alpha^{N-1}\right)(N-1)\delta}{\alpha^{N-1}}. \tag{12}$$

The proof of Theorem 4 is presented in the Appendix. Note that when the noise $m_i(t)$ can be viewed as the dynamic changes of node i's state at time t, (3) is a dynamic average consensus system similar to the one considered in [22]. Thus, by referring to [22, Th. 3.4], we obtain a conservative bound (12). According to (4)

$$\eta = \frac{\left(1+\alpha^{N-1}\right)(N-1)\delta}{\alpha^{N-1}}. \tag{13}$$

Note that (13) gives a closed-form expression revealing the relationship between the noise bound and the consensus accuracy of the discrete average consensus (3). The result can be applied to the worst connectivity and does not depend on the underlying topologies. It follows from (13) that (3) achieves better consensus accuracy when δ becomes smaller.

It is also observed from Theorem 4 that bounded noise has a nonignorable influence on the consensus accuracy, as the term, α^{N-1}, is included in the denominator of (13). It mainly results from the asymptotic convergence property of the average consensus, which means that slow convergence speed causes error accumulation. In addition, if the noise at each node is positive, i.e., at each iteration a positive value is injected into the network, the states of the nodes will drift to infinity and will hence not converge. An algorithm with fast convergence can then mitigate the accumulation of errors, thus achieving consensus with higher accuracy. It motivates us to design an algorithm to realize consensus with higher accuracy.

IV. Fast Consensus Under Bounded Noise

In this section, a novel algorithm (Algorithm 1) named FCBN is designed to realize consensus with higher accuracy. FCBN has two main features. First, each node updates its own value by referring to the core idea of maximum-value-based consensus. Second, the maximum value after initialization is utilized to limit the updating so as to prevent the nodes from drifting to infinity.

Before giving the description of the algorithm, we first provide some notations. Let $H_i(k)$ denote the stored value at iteration k for node i, where $i \in V$. Let secondmax$\{\cdot\}$

Algorithm 1 FCBN

Initialization

1. Each node i initializes its state $x_i(0), i \in V$ and sends out its broadcast value, $x_i^+(0) = x_i(0) + m_i(0)$, to its neighbors.
2. After receiving all neighbor nodes' broadcast values, node i updates its own value as follows
   a) $x_i(0) = \text{secondmax}\{x_i(0), x_j^+(0), j \in \mathcal{N}_i(0)\}$
   b) $H_i(0) = \max\{x_j^+(0), j \in \mathcal{N}_i(0)\}$

Iteration

3. At iteration $k+1$, node i broadcasts $\{x_i^+(k), H_i(k)\}$ to its neighbor nodes, where $x_i^+(k) = x_i(k) + m_i(k), k \geq 0$.
4. Based on the information sets from its neighbors, node i updates its state as follows.
   a) If $\max\{H_i(k), H_j(k), j \in \mathcal{N}_i(k)\} > \max\{x_j^+(k)\}$, then $x_i(k+1) = \max\{x_i(k), x_j^+(k)\}$, $H_i(k+1) = \max\{x_j^+(k)\}$ for $j \in \mathcal{N}_i(k)$;
   b) otherwise, $x_i(k+1) = \max\{x_i(k), H_i(k), H_j(k)\}$, $H_i(k+1) = \max\{x_i(k), H_i(k), H_j(k)\}$ for $j \in \mathcal{N}_i(k)$.

denote the second largest value in the set of $\{\cdot\}$, e.g., $\text{secondmax}\{1, 2, 3\} = 2$. $x_{im}^+(k) = \max\{x_j^+(k), j \in \mathcal{N}_i\}$, $H_{im}(k) = \max\{H_i(k), H_j(k), j \in \mathcal{N}_i\}$.

For FCBN algorithm, according to the initialization, each node obtains and stores the maximum value of its neighbors' as an upper bound to keep every node's state not larger than the upper bounded after each updating. Meanwhile, each node uses the second-maximum value for updating. Thus, after initialization, all nodes will be constrained by the upper bound stored by them. Hereafter, at each iteration, each node updates its own value based on the maximum of its neighbors', and thus the updated value is not larger than the upper bound stored by itself at this iteration. Note that each updating is similar to the maximum-value-based consensus [3], thus, FCBN can have a fast convergence speed and decrease the accumulated error caused by the noise. It is noted that FCBN is different from maximum time synchronization (MTS) proposed in [3]. For MTS, each node utilizes the maximum value of its neighbors' as its updating value at each iteration, thus making the ultimate value of each node asymptotically approach to the maximum initial value of the whole network when the noise is ignored. However, if bounded noise exists, the maximum value of the whole network will increase each iteration, thus the convergence cannot be guaranteed with MTS. For FCBN, an $H_i(t)$, which is bounded by the maximum initial value, is designed to bound the updating value of each node i at each iteration. Then, we have the following result, which reveals a max–min difference of the steady states of all the nodes.

Theorem 5: Assume G is time-invariant and connected. By FCBN, we have

$$|x_i(k) - \gamma| \leq \frac{d\delta}{2}, \forall i \in V \tag{14}$$

for $\forall k \geq d$, where k denotes the kth iteration of the algorithm, d is the diameter of the network, and γ is a constant and equals to the maximum value among nodes' states after initialization of the algorithm.

Proof: After initialization, assume node m has the maximum state, i.e., $x_m(0) \geq x_i(0), i \in V$. Note that all nodes will use the second maximum value among their neighbors as their states and the maximum value as the to-be-forwarded value $H_i(0)$ for each $i \in V$, which leads to $H_m(0) \geq x_m(0)$.

It follows from step 4 that:

$$x_i(k+1) = \max\left\{x_i(k), \min\left\{x_{im}^+(k), H_{im}(k)\right\}\right\} \tag{15}$$

for $k \geq 0$ and

$$H_i(k+1) = \min\left\{x_{im}^+(k), \max\{x_i(k), H_{im}(k)\}\right\} \tag{16}$$

for $\forall i \in V$. Thus

$$x_i(k) \leq x_i(k+1) \leq \max\{x_i(k), H_{im}(k)\}, i \in V \tag{17}$$

which means each $x_i(k)$ is an increasing function, and

$$H_i(k+1) \leq \min\left\{x_{im}^+(k), H_{im}(k)\right\} \leq \max\left\{H_j(k), j \in V\right\}$$

which gives that

$$\max\left\{H_j(k+1), j \in V\right\} \leq \max\left\{H_j(k), j \in V\right\} \tag{18}$$

i.e., $\max\{H_j(k), j \in V\}$ is a decreasing function. Combining (17) and (18), we obtain

$$x_i(0) \leq \cdots \leq x_i(k) \leq \max\left\{x_j(k), H_j(k), j \in V\right\} \tag{19}$$

and

$$\begin{aligned}\max\{x_i(k), H_i(k), i \in V\} &\leq \max\{x_i(k-1), H_i(k-1), i \in V\} \\ &\leq \cdots \leq \max\{x_i(1), H_i(1), i \in V\}.\end{aligned} \tag{20}$$

Combining (19) and (20), one can obtain that

$$\begin{aligned}x_i(0) \leq \cdots \leq x_i(k) &\leq \max\left\{x_j(k), H_j(k), j \in V\right\} \\ &\leq \cdots \leq \max\left\{x_j(1), H_j(1), j \in V\right\}, i \in V.\end{aligned}$$

Then, it follows from (15) and (16) that:

$$x_i(1) \leq \max\left\{x_i(0), x_{im}^+(0)\right\} \leq x_m(0) + \frac{\delta}{2}$$

and $H_i(1) \leq x_{im}^+(0) \leq x_m(0) + (\delta/2)$, i.e., all $x_i(1)$ and $H_i(1)$ are no more than $x_m(0) + (\delta/2)$. From (18) and (20), we have

$$x_i(0) \leq x_i(k) \leq x_m(0) + \frac{\delta}{2}, i \in V \tag{21}$$

$$\begin{aligned}H_i(k) &\leq \max\left\{H_j(k), j \in V\right\} \leq \max\left\{H_j(1), j \in V\right\} \\ &\leq x_{im}^+(0) \leq x_m(0) + \frac{\delta}{2}, i \in V\end{aligned} \tag{22}$$

for $k \geq 1$. Thus, for node m, one infers that $x_m(0) \leq x_m(k) \leq x_m(0) + (\delta/2)$ and $H_m(k) \leq x_m(0) + (\delta/2)$ for $k \geq 0$.

At iteration 1, for the one-hop neighbor node m_1 of node m, it follows from (15) that:

$$x_{m_1}(1) \geq \min\{x_m^+(0), H_m(0)\} \geq x_m(0) - \frac{\delta}{2}$$

and from (16) that

$$H_{m_1}(1) \geq \min\{x_m^+(0), H_m(0)\} \geq x_m(0) - \frac{\delta}{2}.$$

Note that each $x_i(k)$ is an increasing function of iteration k and satisfies (21), which means that

$$x_m(0) + \frac{\delta}{2} \geq x_{m_1}(1) \geq x_m(0) - \frac{\delta}{2}, k \geq 1.$$

Meanwhile, each $H_i(k)$ satisfies (22)

$$x_m(0) + \frac{\delta}{2} \geq H_{m_1}(1) \geq x_m(0) - \frac{\delta}{2}, k \geq 1.$$

At iteration 2, for the two-hop neighbor node m_2 of node m, it follows from (15) and (16) that:

$$x_{m_2}(2) \geq \min\left\{x_{m_1}^{+}(1), H_{m_1}(1)\right\} \geq x_m(0) - \frac{2 \times \delta}{2}$$
$$H_{m_2}(2) \geq \min\left\{x_{m_1}^{+}(1), H_{m_1}(1)\right\} \geq x_m(0) - \frac{2 \times \delta}{2}.$$

It still follows from (21) and (22) that:

$$x_m(0) + \frac{\delta}{2} \geq x_{m_2}(2) \geq x_m(0) - \frac{2 \times \delta}{2}, k \geq 1$$
$$x_m(0) + \frac{\delta}{2} \geq H_{m_2}(2) \geq x_m(0) - \frac{2 \times \delta}{2}, k \geq 1.$$

Similarly, at iteration l, for each l-hop neighbor node m_l of node m, we can infer from (15) and (16) that

$$x_{m_l}(l) \geq \min\left\{x_{m_{l-1}}^{+}(l-1), H_{m_{l-1}}(l-1)\right\} \geq x_m(0) - \frac{l \times \delta}{2}$$
$$H_{m_l}(l) \geq \min\left\{x_{m_{l-1}}^{+}(l-1), H_{m_{l-1}}(l-1)\right\} \geq x_m(0) - \frac{l \times \delta}{2}.$$

Thus

$$x_m(0) + \frac{\delta}{2} \geq x_{m_l}(l) \geq x_m(0) - \frac{l \times \delta}{2}, k \geq l$$
$$x_m(0) + \frac{\delta}{2} \geq H_{m_l}(l) \geq x_m(0) - \frac{l \times \delta}{2}, k \geq 1.$$

Therefore, we have

$$x_m(0) + \frac{\delta}{2} \geq x_i(d) \geq x_m(0) - \frac{d \times \delta}{2}$$
$$x_m(0) + \frac{\delta}{2} \geq H_i(d) \geq x_m(0) - \frac{d \times \delta}{2}$$

where d is the diameter of the network.

At iteration $d+1$, according to step 4, if

$$\max\left\{H_i(d), H_j(d), j \in \mathcal{N}_i\right\} > \max\left\{x_j^{+}(d), j \in \mathcal{N}_i\right\}$$

we have

$$x_i(d+1) = \max\left\{x_i(d), x_j^{+}(d)\right\}$$
$$\in \left[x_m(0) + \frac{\delta}{2}, x_m(0) - \frac{d \times \delta}{2}\right]$$
$$H_i(d+1) = \max\left\{x_j^{+}(d)\right\} < \max\left\{H_i(d), H_j(d)\right\}$$
$$\in \left[x_m(0) + \frac{\delta}{2}, x_m(0) - \frac{d \times \delta}{2}\right]$$

where $j \in \mathcal{N}_i$. Otherwise, for $\forall i \in V$, we will obtain

$$x_i(d+1) = \max\left\{x_i(d), H_i(d), H_j(d), j \in \mathcal{N}_i\right\}$$
$$\in \left[x_m(0) + \frac{\delta}{2}, x_m(0) - \frac{d \times \delta}{2}\right]$$
$$H_i(d+1) = \max\left\{x_i(d), H_i(d), H_j(d), j \in \mathcal{N}_i(k)\right\}$$
$$\in \left[x_m(0) + \frac{\delta}{2}, x_m(0) - \frac{d \times \delta}{2}\right].$$

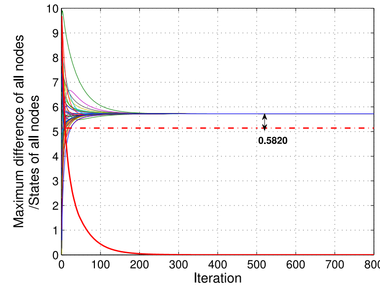


Fig. 1. Verification for the sufficient condition.

That is, all nodes will have

$$x_m(0) + \frac{\delta}{2} \geq x_i(k) \geq x_m(0) - \frac{d \times \delta}{2}, k \geq d$$
$$x_m(0) + \frac{\delta}{2} \geq H_i(k) \geq x_m(0) - \frac{d \times \delta}{2}, k \geq d.$$

Therefore, it is concluded from the above deduction that

$$|x_i(t) - x_m(0)| \leq \frac{d\delta}{2}, i \in V. \quad \blacksquare$$

In the above theorem, the topology is assumed to be fixed for simplicity of description. For dynamic topology we can use similar methodology to prove the theorem, and more discussions will be presented in the simulation section. From Theorem 5, one infers that by FCBN, we have

$$0 \leq \lim_{t\to\infty} D_t \leq d\delta \leq (N-1)\delta \tag{23}$$

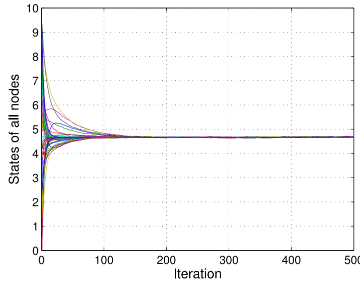
which obtains a more tight value of D_t, comparing with (12). And, one also infers that $\lim_{t\to\infty} |(1/N)\sum_{i=1}^{N} x_i(t) - \bar{x}| \leq |\gamma - \bar{x}|$.

Note that the upper bound stored at each node, its value at each iteration is not larger than that during the last iteration in FCBN. Therefore, after each iteration, FCBN guarantees that the updated value of each node is nondecreasing but will not be larger than its stored value, while the stored value is nonincreasing but will not be smaller than the updated value. Ultimately, the value of each node's state will not increase or decrease with the noise, which means that the consensus will not drift with time, and it even provides a positive possibility for the algorithm to achieve consensus completely.

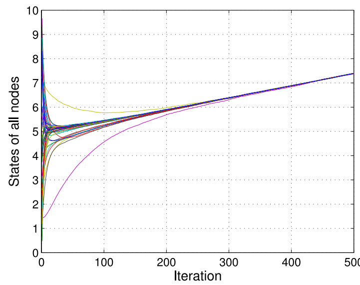
V. Performance Evaluation

In this section, extensive simulations are conducted to verify the results. A fixed topology with $N = 50$ is first considered in the following simulations, where the nodes are randomly deployed over a 100 $m * 100$ m 2-D square area and the communication range between nodes is 20 m. The initial value of each node $x_i(0)$ is randomly selected from (0, 10). Bounded noise with bound δ is generated by randomly choosing the value of sine function which is amplified by δ. For the algorithm DACA, each node i for $i \in V$ assigns its neighbor $j \in \mathcal{N}_i$ a weight $a_{ij} = (1/N)$ and a weight $1 - \sum_{j\in\mathcal{N}_i} a_{ij}$ for itself, and thus has $\alpha = (1/N)$.

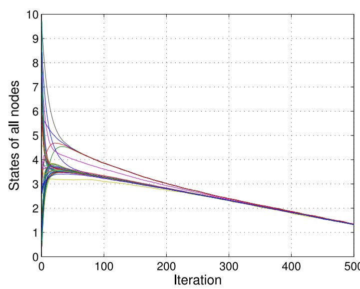
We illustrate the sufficient condition in Theorem 2. Specifically, we have bounded noises $e^{-(t+1)}$ to the nodes. Fig. 1 shows that all the nodes achieve discrete consensus in



(a)

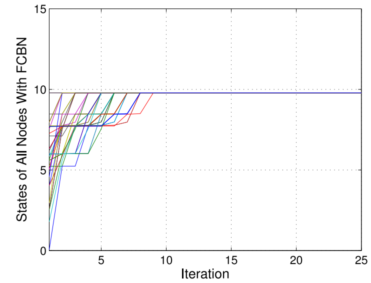


(b)

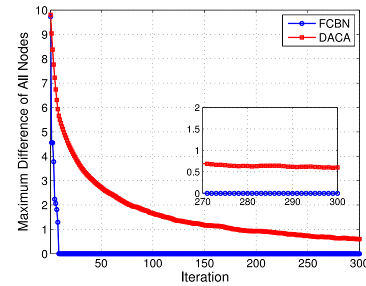


(c)

Fig. 2. Consensus under DACA with bounded noise for a fixed topology. (a) Bounded noise $|m_i(t)| \le (1/2)\delta$. (b) Bounded noise $0 \le m_i(t) \le (1/2)\delta$. (c) Bounded noise $-(1/2)\delta \le m_i(t) \le 0$.

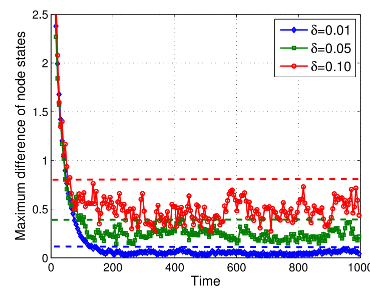


(a)

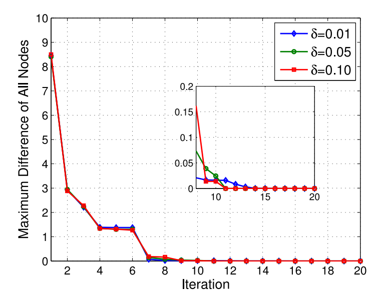


(b)

Fig. 3. FCBN for a fixed topology. (a) Evolution of the states with FCBN. (b) Performance comparison.



(a)



(b)

Fig. 4. Evolution of D_t under different noise bounds for a fixed topology. (a) D_t with DACA. (b) D_t with FCBN.

the presence of the bounded noises $e^{-(t+1)}$, which is indicated by the decreasing red line in the figure. Moreover, it is revealed by the results that a gap $g = 0.5820$ is acquired under the bounded noise $e^{-(t+1)}$, which is in accordance with Theorem 3. From (11), we have

$$g \le \lim_{t\to\infty} \sum_{p=0}^{t} \max_{i\in V} |m_i(p)| \le \lim_{t\to\infty} \sum_{p=0}^{t} e^{-(p+1)} = 0.5820$$

which is a much tighter gap than the upper bound BN $= 29.1$.

Then, for DACA, consider the relationship between the consensus accuracy and the noise bound. It is first observed from Fig. 2 that the discrete average consensus system is able to reach a state that all nodes fluctuate within a small bound under three types of bounded noises. However, it is notable that when the noise is always positive (or negative) the stable state will become larger (or smaller), which is undesirable in applications. Noises with three different bounds, $\delta = 0.01, 0.05$, and 0.1, are imposed on the discrete system. Convergence accuracy decreases with the increasing of the noise bound δ as shown in Fig. 4(a), and is always within the bound given by Theorem 4. However, it is also apparent from Fig. 2(b) and (c) that the stable state may drift with time, meaning that the stable states of all nodes will increase or decrease and may even drift to infinity. Furthermore, it can be observed from the figures that all nodes fluctuate within a bound. The results illustrate that DACA has bad consensus accuracy under bounded noise.

Next, we consider the performance of FCBN. From Fig. 3(a) and (b), it is observed that all nodes achieve convergence within 50 iterations shown by the blue line, illustrating that FCBN has a linear convergence speed. Compared with the performance of DACA shown by the red line, which realizes

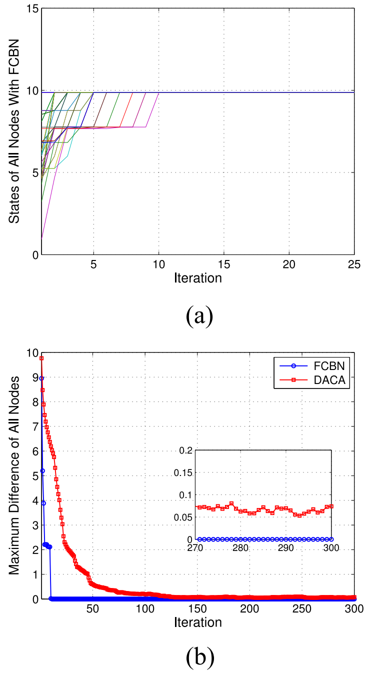


Fig. 5. Performance under FCBN for a dynamic topology. (a) Evolution of the states with FCBN. (b) Performance comparison.

asymptotic convergence, FCBN has a faster convergence rate. And it is revealed by the maximum difference of all nodes for FCBN and DACA that FCBN has smaller maximum difference, meaning that FCBN is able to acquire higher consensus accuracy. Fig. 4(b) shows the consensus accuracy under different noise bounds $\delta = 0.01, 0.05$, and 0.1 for FCBN. Clearly, FCBN achieves fast convergence under different noise bounds with high consensus accuracy. Moreover, FCBN is even able to realize complete convergence. Thus, FCBN not only has faster convergence rate but also higher consensus accuracy, obtained from comparing Fig. 4(b) with Fig. 4(a).

We also simulate the performance of FCBN under a dynamic topology, where two nodes move randomly at each iteration. Fig. 5(a) shows the states of all nodes with FCBN under the dynamic topology, from which one notes that all nodes converge within 50 iterations. Fig. 5(b) illustrates the performance comparison between FCBN and DACA under the dynamic topology, which shows that FCBN still has a faster convergence rate and has a higher consensus accuracy than DACA. Thus, FCBN is robust to the topology changes.

VI. Conclusion

We investigate the discrete average consensus problem in the presence of bounded noises. We obtain a necessary condition and a sufficient condition to guarantee the complete convergence of discrete consensus, and derive an explicit expression connecting the noise bound and the consensus accuracy. We further propose a novel algorithm FCBN to reduce the effect of the bounded noise and accelerate the convergence speed. Simulation results demonstrate that the consensus accuracy decreases as the noise bound increases. It also shows that FCBN has a faster convergence speed and a higher consensus accuracy than those of DACA.

Appendix
Proof of Theorem 3.4

Proof: Since $\sum_{j=1}^{N} a_{ij}(0) = 1$, for every node i

$$x_i(1) = \sum_{j=1}^{N} a_{ij}(0)x_j(0) + \sum_{j=1}^{N} a_{ij}(0)m_j(0) \geq \sum_{j=1}^{N} a_{ij}(0)h_0 + \sum_{j=1}^{N} a_{ij}(0)m_0 = h_0 + m_0. \tag{24}$$

Since (24) holds for all $i \in V$

$$h_1 \geq h_0 + m_0. \tag{25}$$

Repeatedly applying (25), one has $h_t \geq h_0 + \sum_{q=0}^{t-1} m_q$. Then

$$\begin{aligned} x_i(t+1) - h_0 - \sum_{q=0}^{t} m_q &= \sum_{j=1}^{N} a_{ij}(t)\left(x_j(t) - h_0 - \sum_{q=0}^{t-1} m_q\right) + \sum_{j=1}^{N} a_{ij}(t)m_j(t) - m_t \\ &\geq a_{ii}(t)\left(x_i(t) - h_0 - \sum_{q=0}^{t-1} m_q\right) \\ &\geq \alpha\left(x_i(t) - h_0 - \sum_{q=0}^{t-1} m_q\right). \end{aligned}$$

For those $t \in [0, K]$

$$\begin{aligned} x_i(t) - h_0 - \sum_{q=0}^{t-1} m_q &\geq \alpha^{t-1}(x_i(1) - h_0 - m_0) \\ &\geq \alpha^{t}(x_i(0) - h_0) \geq \alpha^{N-1}(x_i(0) - h_0) \end{aligned}$$

for node i.

For those $s \in \mathcal{D}_1$, where $\mathcal{D}_1$ has been defined in Section II, one notes that

$$\begin{aligned} x_s(1) - h_0 - m_0 &= \sum_{j=1}^{N} a_{sj}(0)\big(x_j(0) - h_0\big) + \sum_{j=1}^{N} a_{sj}(0)m_j(0) - m_0 \\ &\geq a_{si}(t)(x_i(0) - h_0) \geq \alpha(x_i(0) - h_0) = \beta_1(x_i(0) - h_0) \end{aligned}$$

where $\beta_1 = \alpha$.

For those $j \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{K-1}$ and communicating with node in $\mathcal{D}_K$, $K \leq N-1$, we have

$$x_j(K-1) - h_0 - \sum_{q=0}^{K-2} m_q \geq \beta_{K-1}(x_i(0) - h_0).$$

We now discuss those $s \in \mathcal{D}_K$. Hence, we have

$$\begin{aligned}
x_s(K) - h_0 - \sum_{q=0}^{K-1} m_q
&= \sum_{j=1}^{N} a_{sj}(K-1)\left(x_j(K-1) - h_0 - \sum_{q=0}^{K-2} m_q\right) \\
&\quad + \sum_{j=1}^{N} a_{sj}(K-1)m_j(K-1) - m_{K-1} \\
&\geq a_{sj}(K-1)\left(x_j(K-1) - h_0 - \sum_{q=0}^{K-2} m_q\right) \\
&\geq \alpha\left(x_j(K-1) - h_0 - \sum_{q=0}^{K-2} m_q\right) \\
&\geq \alpha\beta_{K-1}(x_i(0) - h_0).
\end{aligned}$$

Let $\beta_K = \alpha\beta_{K-1}$, then $\beta_K = \alpha^{K-1}\beta_1 = \alpha^K$. Therefore

$$x_j(t) - h_0 - \sum_{q=0}^{K-1} m_q \geq \alpha^K(x_i(0) - h_0) \geq \alpha^{N-1}(x_i(0) - h_0)$$

i.e., $x_j(t) \geq h_0 + \sum_{q=0}^{K-1} m_q + \alpha^{N-1}(x_i(0) - h_0)$, for $t = K$ and node $j \in \mathcal{D}_K$.

Similarly, we can prove that

$$x_j(t) \leq H_0 + \sum_{q=0}^{K-1} M_q - \alpha^{N-1}(H_0 - x_i(0))$$

for $t = K$ and node $j \in \mathcal{D}_K$.

Consequently

$$h_K \geq h_0 + \sum_{q=0}^{K-1} m_q + \alpha^{N-1}(x_i(0) - h_0) \tag{26}$$

$$H_K \leq H_0 + \sum_{q=0}^{K-1} M_q - \alpha^{N-1}(H_0 - x_i(0)). \tag{27}$$

Subtracting (26) from (27), one has $D_K \leq (1 - \alpha^{N-1})D_0 + \sum_{q=0}^{K-1} \Delta m_q$. Thus, we have

$$\begin{aligned}
D_{nK} &\leq \left(1 - \alpha^{N-1}\right)D_{(n-1)k} + \sum_{q=(n-1)k}^{nK-1} \Delta m_q \\
&\leq \left(1 - \alpha^{N-1}\right)^n D_0 + \sum_{q=(n-1)k}^{nK-1} \Delta m_q \\
&\quad + \left(1 - \alpha^{N-1}\right)\sum_{q=(n-2)k}^{(n-1)K-1} \Delta m_q + \cdots + \left(1 - \alpha^{N-1}\right)^n \sum_{q=0}^{K-1} \Delta m_q.
\end{aligned}$$

From (25), $D_1 \geq D_0 + \Delta m_0$. Assuming that the system converges at t, we find an integer $z = \lfloor (t/N-1) \rfloor$ such that

$$\begin{aligned}
D_t \leq D_{z(N-1)} + \sum_{q=z(N-1)}^{t-1} \Delta m_q &\leq \left(1 - \alpha^{N-1}\right)^z D_0 \\
&\quad + \sum_{q=z(N-1)}^{t-1} \Delta m_q + \Phi(z)
\end{aligned}$$

where

$$\Phi(z) = \sum_{q=(z-1)(N-1)}^{z(N-1)-1} \Delta m_q + \cdots + \left(1 - \alpha^{N-1}\right)^z \sum_{q=0}^{N-1} \Delta m_q.$$

Since $|\Delta m_t| \leq \delta$ and $(1 - \alpha^{N-1})^z D_0 = 0$ when $t \to \infty$, we have $\Phi(z) \leq ((N-1)\delta/\alpha^{N-1})$ and $\sum_{q=z(N-1)}^{t-1} \Delta m_q \leq (N-1)\delta$. Consequently, the stable state of the nodes will fluctuate within the bound $\theta = (1 + \alpha^{N-1}/\alpha^{N-1})(N-1)\delta$. For all $t \geq 0$, we have

$$\begin{aligned}
\sum_{i=1}^{N} x_i(t+1) &= \mathbf{1}^T \mathbf{A}(t)(\mathbf{X}(t) + \mathbf{M}(t)) \\
&= \sum_{i=1}^{N} x_i(t) + \sum_{i=1}^{N} m_i(t) = \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}\sum_{q=0}^{t} m_i(q).
\end{aligned}$$

Since $h_t \leq x_i(t) \leq H_t$ and $h_t \leq (1/N)\sum_{i=1}^{N} x_i(0) + (1/N)\sum_{i=1}^{N}\sum_{q=0}^{t} m_i(q) \leq H_t$, we obtain

$$\lim_{t\to\infty}\left| x_i - \frac{1}{N}\sum_{i=1}^{N} x_i(0) - \frac{1}{N}\sum_{i=1}^{N}\sum_{q=0}^{t-1} m_i(q)\right| \leq \lim_{t\to\infty} D_t \leq \theta.$$

■

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